Music Information Geometry

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Outline

1. Introduction
2. Information geometry
3. Proposed framework
4. Obtained results
5. Conclusion
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1. Introduction
   - A bit of history about science and music
   - Motivations towards information geometry

2. Information geometry

3. Proposed framework

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5. Conclusion
Where do we come from?

- Pythagoras (~ 570–495 BC): relation between string length and produced sound, Pythagorean tuning.

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- But also indirectly Fourier, Shannon, etc.
What do we need?

**Figure:** Levels of representation of audio, waveform and spectrogram representations.
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- Develop a comprehensive framework that allows to quantify, process and represent the information contained in audio signals.
- Fill in the gap between signal and symbolic representations.

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   - Background
   - Exponential families

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What is information geometry?

Statistical differentiable manifold.

Under certain assumptions, a parametric statistical model \( S = \{ p_\xi : \xi \in \Xi \} \) of probability distributions defined on \( X \) forms a differentiable manifold.
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- Example: $p_\xi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}$ for all $x \in \mathcal{X} = \mathbb{R}$, with $\xi = [\mu, \sigma^2] \in \Xi = \mathbb{R} \times \mathbb{R}_{++}$. 
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Fisher information metric [Rao, 1945, Chentsov, 1982].

Under certain assumptions, the Fisher information matrix defines the unique Riemannian metric $g$ on $S$: $g_{ij}(\xi) = \int_{x \in \mathcal{X}} \partial_i \log p_\xi(x) \cdot \partial_j \log p_\xi(x) \cdot p_\xi(x) \cdot dx$. 

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**Dual affine connections** [Chentsov, 1982, Amari & Nagaoka, 2000].

Under certain assumptions, there is a unique family of dual affine connections \( \{ \nabla^{(\alpha)}, \nabla^{(-\alpha)} \}_{\alpha \in \mathbb{R}} \) on \( (S, g) \) called \( \alpha \)-connections.
How to use information geometry from a computational viewpoint?

**Exponential family.**

$$p_\theta(x) = \exp \left( \theta^T T(x) - F(\theta) + C(x) \right) \text{ for all } x \in \mathcal{X}.$$ 

- $\theta$: natural parameters, vector belonging to a convex open set $\Theta$.
- $F$: log-normalizer, real-valued, strictly convex smooth function on $\Theta$.
- $C$: carrier measure, real-valued function on $\mathcal{X}$.
- $T$: sufficient statistic, vector-valued function on $\mathcal{X}$. 

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- Dually flat geometry, Hessian structure \( (g = \nabla^2 F) \), generated by the potential \( F \) together with its conjugate potential \( F^* \) defined by the Legendre-Fenchel transform: \( F^*(\eta) = \sup_{\theta \in \Theta} \theta^T \eta - F(\theta) \), which verifies \( \nabla F^* = (\nabla F)^{-1} \) so that \( \theta = \nabla F^*(\eta) \).
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- Generalizes the self-dual Euclidean geometry, with notably two canonically associated Bregman divergences \( B_F \) and \( B_{F^*} \) instead of the self-dual Euclidean distance, but also dual geodesics, a generalized Pythagorean theorem and dual projections.
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**Bregman divergence.**

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- Canonical divergences of dually flat spaces, “bijection” with exponential families [Amari & Nagaoka, 2000, Banerjee et al., 2005]:
  \[ D_{KL}(p_{\xi} \parallel p_{\xi'}) = B_F(\theta' \parallel \theta) = B_{F^*}(\eta \parallel \eta'). \]
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  - Centroid computation and hard clustering \((k\text{-means}).\)
  - Parameter estimation and soft clustering (expectation-maximization).
  - Proximity queries in ball trees (nearest-neighbors and range search).
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   - General architecture
   - Sound descriptors modeling
   - Temporal modeling
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How to design an audio system based on information geometry?

- **Scheme:**
  1. Represent the incoming audio stream with short-time sound descriptors $d_j$.
  2. Model these descriptors as probability distributions $p_{\theta_j}$ from a given exponential family.
  3. Use the framework of computational information geometry on these distributions.

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Important need for temporal modeling.

Potential applications [Cont et al., 2011]:
- Audio content analysis.
- Segmentation of audio streams.
- Automatic structure discovery of audio signals.
- Sound processing and synthesis.

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How to model sounds?

- Computation of a sound descriptor $d_j$:
  - Fourier or constant-Q transforms for information on the spectral content.
  - Mel-frequency cepstral coefficients for information on the timbre.
  - Many other possibilities.

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How to model sounds?

- Computation of a sound descriptor $d_j$:
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  - Mel-frequency cepstral coefficients for information on the timbre.
  - Many other possibilities.
- Modeling with a probability distribution $p_{\theta_j}$ from an exponential family:
  - Categorical distributions.
  - Many other possibilities.

**Figure:** Sound descriptors modeling.
How to take time into account?

- Model formation: from signal to symbol.
  - Assumption of quasi-stationary audio chunks.
  - Change detection adapted from CuSum [Basseville & Nikiforov, 1993].

Figure: Model formation at time $t$. 
How to take time into account?

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  \[ \text{Figure: Model formation at time } t. \]

- **Factor oracle: from symbol to syntax (and from genetics to music!).**
  - Forward transitions: original sequence factors.
  - Backward links: suffix relations, common context.

  \[ \text{Figure: Factor oracle of the word } abbbaab. \]
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   - Music similarity analysis
   - Musical structure discovery
   - Query by similarity
   - Audio recombination by concatenative synthesis
   - Computer-assisted improvisation
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Audio segmentation

**Figure:** Segmentation of the *1st Piano Sonate, 1st Movement, 1st Theme*, Beethoven.
Music similarity analysis

Figure: Similarity analysis of the 1st Piano Sonate, 3rd Movement, Beethoven.
Musical structure discovery

Figure: Structure discovery of the *1st Piano Sonate, 3rd Movement*, Beethoven.
Query by similarity

Figure: Query by similarity of the 1st Theme over the entire 1st Piano Sonate, 1st Movement, Beethoven.
Audio recombination by concatenative synthesis

Figure: Audio recombination of African drums by concatenative synthesis of congas.
Computer-assisted improvisation

**Figure**: Computer-assisted improvisation, Fabrizio Cassol and Philippe Leclerc.
What we (don’t) have

Summary and perspectives.

- Representations.
- Descriptors modeling.
- Temporal modeling.
- Temporality of events.
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- Many possibilities.
- Combinations of descriptors.
- Complex representations.
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Summary and perspectives.

- Representations.
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- Exponential families and Bregman divergences.
- Mixture models of a given exponential family.
- Other geometries, divergences, metrics.
Summary and perspectives.

- Representations.
- Descriptors modeling.
- Temporal modeling.
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- On-line segmentation and factor oracle.
- On-line clustering and equivalence between symbols.
- Overlap between symbols and other temporal models.
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- Assumption of quasi-stationarity.
- Non-stationarity modeling.
- Time series.
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- Resources on IG: http://imtr.ircam.fr/imtr/Music_Information_Geometry
- National research group: IRCAM, Ecole Polytechnique, Thales, etc.
- Brillouin seminar:
  http://www.informationgeometry.org/Seminar/seminarBrillouin.html
- IGAIA 2012.
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- Thank you very much for your attention! Questions?
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Information and accuracy attainable in the estimation of statistical parameters.